Topology Vol. 13, pp. 19–24. Pergamon Press, 1974. Printed in Great Britain

MANIFOLD FACTORS THAT ARE MANIFOLD QUOTIENTS

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(*Received* 9 *January* 1973; *revised* 23 *July* 1973)

IN THIS paper we prove the following theorem.

THEOREM 1. *Suppose that X is a topological space such that*

(a) *$X \times \mathbb{R}^k$ is a smooth (PL) $(n + k)$-manifold without boundary ($n \geq 5$), and*

(b) *X is locally euclidean except at points of a 0-dimensional set K.*

Then there exist a smooth (PL) n-manifold M and a proper cell-like map $g: M \to X$ such that $g|g^{-1}(X - K)$ is a homeomorphism.

This theorem is essentially the converse to a conjecture that has been of interest to a number of topologists in recent years—primarily because of its relation to the elusive "double suspension problem" (cf. [3] and [12].) The proof depends heavily upon the Product Structure Theorem of Kirby–Siebenmann [5] and the main results of Siebenmann's thesis [11]. The first author would like to express his appreciation to Dennis Sullivan for many profitable conversations.

Throughout this paper we shall restrict ourselves to the following hypotheses:

(1) X is a topological space such that $X \times \mathbb{R}^k$ is a smooth $(n + k)$-manifold without boundary.

(2) $n \geq 5$, and

(3) X is locally euclidean except at points of a 0-dimensional set K (which is necessarily closed in X).

Recall that a compact set C is *cell-like* if it can be embedded in a manifold as a cellular set. This is equivalent to C having property UV^∞ (cf. [7]). Observe that X is locally compact and locally contractible.

We shall deal only with the smooth case, since the proof in the *PL* case is similar.

Our sole requirement of Kirby–Siebenmann is in obtaining a smooth structure for $X - K$.

PROPOSITION 1. *$X - K$ has a smooth structure compatible with the smooth structure on $(X - K) \times \mathbb{R}^k$ (as an open subset of the smooth manifold $X \times \mathbb{R}^k$).*

Proof. This is a direct application of the Product Structure Theorem of [5].

† The first author was partially supported by National Science Foundation Grant GP19964.

PROPOSITION 2. *Let N' be a closed neighborhood of K in X with compact components each having a smooth $(n-1)$-manifold as boundary. Let N be a component of N'. Then N has the homotopy type of a finite cell complex.*

Proof. Suppose that N is given as above. Since $K \cap N$ is a compact, 0-dimensional subset of Int N, there exists for each $\varepsilon > 0$ a finite cover $U_1, \ldots, U_m$ of $K \cap N$ in N such that $\mathrm{Cl}U_i \cap \mathrm{Cl}U_j = \phi$ if $i \neq j$ and diam $U_i < \varepsilon$ for $i = 1, \ldots, m$. Thus, since N is locally contractible, there exists a compact neighborhood P of $K \cap N$ with smooth boundary such that each component of P is null-homotopic in N. Then $\mathrm{Cl}(N - P)$ has the structure of a finite complex ([10, Theorem 10.6]). Now P, as well as N, is dominated by a finite complex, since each is a compact ANR. These two facts imply that $i_* : \tilde{K}_0(\pi_1 P) \to \tilde{K}_0(\pi_1 N)$ takes Wall's obstruction to finiteness [13] of P to the obstruction to finiteness of N (Theorem 6.5 of [11]). But $i_* = 0$.

PROPOSITION 3. *Suppose that A and B are finite cell complexes such that A deformation retracts to B and for each component L of* $\mathrm{Cl}(A - B)$, $\mathrm{Wh}(G_L) = 0$, *where* $G_L = \mathrm{im}(\pi_1 L \to \pi_1 A)$. *Then the torsion* $\tau(A, B) = 0$.

Proof. In the statement of the proposition, $\mathrm{Wh}(G)$ denotes the Whitehead group of a group G (cf. [9]). The proof of the proposition follows closely that of Lemma 7.2 of [9] and roughly goes as follows.

Let $(\tilde{A}, \tilde{B})$ denote the universal cover of (A, B) with covering map $p: \tilde{A} \to A$. Then $\mathrm{Cl}(\tilde{A} - \tilde{B})$ is the union of the components of $p^{-1}(L)$ as L ranges over the components of $\mathrm{Cl}(A - B)$. By choosing for each component L of $\mathrm{Cl}(A - B)$ a component $\tilde{L}$ of $p^{-1}(L)$, we can obtain generators of the cellular chain complex $C_*(\tilde{A}, \tilde{B})$ of $(\tilde{A}, \tilde{B})$ and for the boundary subcomplex $B_*(\tilde{A}, \tilde{B})$ (as $\mathbb{Z}(\pi_1 A)$-modules) so that the matrix relating the bases c_q and $b_q b_{q-1}$ of each C_q is of the form $\bigoplus_L M_L{}^q$, where $M_L{}^q$ has coefficients in $\mathbb{Z}(G_L) \subset \mathbb{Z}(\pi_1 A)$. $\left(\text{Given matrices } M_1 \text{ and } M_2,\ M_1 \oplus M_2 = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}.\right)$ Since $\mathrm{Wh}(G_L) = 0$ for each L, we have $\tau(A, B) = 0$. (A calculation similar to this may also be found in [6].)

Remark. Let $T^m = S^1 \times \cdots \times S^1$ (m times) denote the m-dimensional torus. Since $T^{k-1} \times \mathrm{R}$ embeds as an open subset of R^k, $(N \times T^{k-1} \times \mathbb{R}, \mathrm{Bd}N \times T^{k-1} \times \mathbb{R})$ has the structure of a smooth manifold pair (as an open subset of $(N \times \mathbb{R}^k, \mathrm{Bd}N \times \mathbb{R}^k)$).

THEOREM 2. *Suppose that N is a compact component of a neighborhood of K such that* BdN *is a smooth $(n-1)$-manifold. Then there exist a smooth n-manifold M with boundary and a diffeomorphism h: $M \times T^{k-1} \times \mathbb{R} \to N \times T^{k-1} \times \mathbb{R}$ whose restriction to the boundary commutes with the projection onto each of the last k factors.*

Proof. Let $N_k \subset N_{k-1} \subset \cdots \subset N_1 \subset N_0 = N$ be compact neighborhoods of $K \cap N$ with smooth boundaries such that each component of N_i is null-homotopic in N_{i-1}.

Consider now the submanifolds $N_k \times T^{k-1} \times \mathbb{R} \subset N_{k-1} \times T^{k-1} \times \mathbb{R} \subset N \times T^{k-1} \times \mathbb{R}$. By Proposition 2, $N_{k-1} \times T^{q-1} \times \mathbb{R}$ satisfies all the necessary conditions required by [11] in order to be expressed as a product $M \times \mathbb{R}$, where M is a smooth $(n+k-1)$-manifold.

The only obstruction lies in $\tilde{K}_0(\pi_1(N_{k-1} \times T^{k-1}))$. (Work with one component of $N_{k-1} \times T^{k-1} \times \mathbb{R}$ at a time.) Observe, however, that $\mathrm{Cl}(N_{k-1} - N_k) \times T^{k-1} \times \mathbb{R}$ is already in this form. Thus, since each component of N_k is null-homotopic in N_{k-1}, Theorem 6.5 of [11] shows that the obstruction in fact lies in $\tilde{K}_0(\mathbb{Z}^{k-1}) = 0$. ($\mathbb{Z}^m$ is the free abelian group on m generators.) Applying [11] we obtain a smooth $(n + k - 1)$-manifold M'_{k-1} and a diffeomorphism $h'_{k-1}\colon M'_{k-1} \times \mathbb{R} \to N_{k-1} \times T^{k-1} \times \mathbb{R}$ such that $h'_{k-1}(\mathrm{Bd}M'_{k-1} \times 0) = \mathrm{Bd}N_{k-1} \times T^{k-1} \times 0$ and, abusing notation slightly, $h'_{k-1}(x, t) = (h'_{k-1}(x), t)$ for $x \in \mathrm{Bd}M'_{k-1}$ and $t \in \mathbb{R}$. Let $M_{k-1} = M'_{k-1} \bigcup_{\mathrm{Bd}M'_{k-1}} \mathrm{Cl}(N - N_{k-1}) \times T^{k-1}$ and extend h'_{k-1} to $h_{k-1}\colon M_{k-1} \times \mathbb{R} \to N \times T^{k-1} \times \mathbb{R}$ via the identity on $\mathrm{Cl}(N - N_{k-1}) \times T^{k-1}$.

Suppose inductively for $i < k$ we have a smooth $(n + k - i)$-manifold M_{k-i}, a smooth $(n + k - i)$-submanifold L_{k-i} of M_{k-i} containing $\mathrm{Bd}M_{k-i}$, a diffeomorphism $g_{k-i}\colon (L_{k-i}, \mathrm{Bd}M_{k-i}) \to (\mathrm{Cl}(N - N_{k-i}) \times T^{k-i}, \mathrm{Bd}N \times T^{k-i})$ and a proper homotopy equivalence

$$h_{k-i}\colon M_{k-i} \times \mathbb{R} \to N \times T^{k-i} \times \mathbb{R}$$

such that

$$h^{-1}_{k+i}(\mathrm{Cl}(N - N_{k-i}) \times T^{k-i} \times \mathbb{R}) = L_{k-i} \times \mathbb{R}$$

and

$$h_{k-i} | L_{k-i} \times \mathbb{R} = g_{k-i} \times \text{identity}.$$

Then we have the following diagram

$$\begin{tikzcd}
\tilde{L}_{k-i} \arrow[r, "\tilde{g}_{k-i}"] \arrow[d, phantom, "\cap"] & \mathrm{Cl}(N - N_{k-i}) \times T^{k-i-1} \times \mathbb{R} \arrow[d, phantom, "\cap"] \\
\tilde{M}_{k-i} \arrow[r, "\tilde{f}"] \arrow[d, "\pi"] & N \times T^{k-i-1} \times \mathbb{R} \arrow[d, "\mathrm{id} \times e"] \\
M_{k-i} \arrow[r, "f"] \arrow[d, "\times 0"] & N \times T^{k-i} = N \times T^{k-i-1} \times S^1 \\
M_{k-i} \times \mathbb{R} \arrow[r, "h_{g-i}"] & N \times T^{k-i} \times \mathbb{R} \arrow[u, "p"]
\end{tikzcd}$$

where p is the projection, f is the indicated composition, e is the exponential map from $\mathbb{R}$ to S^1, $(\tilde{M}_{k-i}, \pi)$ is the induced covering space of M_{k-i}, $\tilde{f}$ lifts f, and $\tilde{g}_{k-i}$ lifts g_{k-i}. Observe that $f | L_{k-i} = g_{k-i}$, f is a homotopy equivalence, $f^{-1}(\mathrm{Cl}(N - N_{k-i}) \times T^{k-i} = L_{k-i}$, $\tilde{f}$ is a *proper* homotopy equivalence, and $\tilde{g}_{k-i}$ is a diffeomorphism.

Set $L_{k-i-1} = \tilde{g}^{-1}_{k-i}(\mathrm{Cl}(N - N_{k-i-1}) \times T^{k-i-1} \times 0)$, $g_{k-i-1} = \tilde{g}_{k-i} | L_{k-i-1}$, $W = \tilde{f}^{-1}(N_{k-i-1} \times T^{k-i-1} \times \mathbb{R})$, and $V = \tilde{f}^{-1}(\mathrm{Cl}(N_{k-i-1} - N_{k-i}) \times T^{k-i-1} \times 0) \subset W$. Then $\tilde{f} | W\colon W \to N_{k-i-1} \times T^{k-i-1} \times \mathbb{R}$ is a proper homotopy equivalence and, hence, W satisfies all the necessary conditions of [11] in order to be a product of a smooth manifold with $\mathbb{R}$. Since each component of N_{k-i} is null-homotopic in N_{k-i-1} and since $\tilde{f}^{-1}(\mathrm{Cl}(N_{k-i-1} - N_{k-i}) \times T^{k-i-1} \times \mathbb{R})$ is already a product over V, the only obstruction to splitting W lies in $\tilde{K}_0(\mathbb{Z}^{k-i-1}) = 0$ [11]. Thus W can be written as a product $M'_{k-i-1} \times \mathbb{R}$ where M'_{k-i-1}

is a smooth $(n+k-i-1)$-manifold and where the product structure coincides with the existing product structure on $\mathrm{Bd}W$. Now set $M_{k-i-1} = L_{k-i-1} \bigcup_{\mathrm{Bd}M_{k-i-1}'} M_{k-i-1}'$ and $h_{k-i-1} = \tilde{f}\colon M_{k-i-1} \times \mathbb{R} \to N \times T^{k-i-1} \times \mathbb{R}$.

We now complete the proof of Theorem 2 by setting $M = M_0$. Clearly each M_{k-i} is diffeomorphic to $M \times T^{k-i}$ so that $M \times T^{k-1} \times \mathbb{R}$ is diffeomorphic to $N \times T^{k-1} \times \mathbb{R}$.

Suppose now that we have N as above. Let $P_2 \subset P_1 \subset N_k \subset \cdots \subset N_1 \subset N_0 = N$ be compact neighborhoods of $N \cap K$ with smooth boundaries such that each component of N_i is null-homotopic in N_{i-1}. Let Q_i $(i = 1, 2)$ be a smooth n-manifold and let $g_i\colon Q_i \times T^{k-1} \times \mathbb{R} \to P_i \times T^{k-1} \times \mathbb{R}$ be a diffeomorphism as given in Theorem 2. In particular g_i induces a diffeomorphism $\mathrm{Bd}Q_i \to \mathrm{Bd}P_i$ $(i = 1, 2)$. Let $M_i = Q_i \bigcup_{\mathrm{Bd}P_i} \mathrm{Cl}(N - P_i)$. Then $M_1 \times T^{k-1} \times \mathbb{R} \cong N \times T^{k-1} \times \mathbb{R} \cong M_2 \times T^{k-1} \times \mathbb{R}$, where "$\cong$" means "is diffeomorphic to", via diffeomorphisms that restrict to the identity on $\mathrm{Cl}(N - P_1) \times T^{k-1} \times \mathbb{R}$.

THEOREM 3. *Given the above data, there exists a diffeomorphism $h\colon M_2 \to M_1$ such that $h|\mathrm{Bd}M_2$ is the identity.*

Proof. Since the proof is very similar to that of Theorem 2, we shall only begin the procedure and leave it to the reader to complete the induction.

Assume without loss of generality that

$$N \times T^{k-1} \times \mathbb{R} = M_1 \times T^{k-1} \times \mathbb{R} = M_2 \times T^{k-1} \times \mathbb{R},$$

where the product structures agree over $\mathrm{Cl}(N - N_k) = \mathrm{Cl}(M_i - N_k)$ $(i = 1, 2)$. Choose $s, t \in \mathbb{R}$ such that $(M_1 \times T^{k-1} \times s) \cap (M_2 \times T^{k-1} \times t) = \phi$. For convenience assume $s = 1$ and $t = 2$. For $j = k, k-1$, let W_j be the closure of the region in $N_j \times T^{k-1} \times \mathbb{R}$ between $(M_1 \times T^{k-1} \times 1)$ and $(M_2 \times T^{k-1} \times 2)$. Then W_j is an h-cobordism between the manifolds $(M_1 \times T^{k-1} \times 1) \cap (N_j \times T^{k-1} \times \mathbb{R})$ and $(M_2 \times T^{k-1} \times 2) \cap (N_j \times T^{k-1} \times \mathbb{R})$. Moreover, $\mathrm{Cl}(W_{k-1} - W_k) = \mathrm{Cl}(N_{k-1} - N_k) \times T^{k-1} \times [1, 2]$, so that W_{k-1} collapses to $W_k \cup [(M_i \times T^{k-1} \times i) \cap (N_{k-1} \times T^{k-1} \times \mathbb{R})]$. Set $B_i = [(M_i \times T^{k-1} \times i) \cap (N_{k-1} \times T^{k-1} \times \mathbb{R})]$ and $A_i = B_i \cup W_k$ $(i = 1, 2)$. Then $\mathrm{Cl}(A_i - B_i) = W_k$. Since each component of N_k is null-homotopic in N_{k-1}, each component L of $\mathrm{Cl}(A_i - B_i)$ has the property that $\mathrm{im}(\pi_1 L \to \pi_1 A) = \mathbb{Z}^{k-1}$ (as in the proof of Theorem 2). Since $\mathrm{Wh}(\mathbb{Z}^{k-1}) = 0$ [2], $\tau(A_i, B_i) = 0$ $(i = 1, 2)$ by Proposition 3. Hence W_{k-1} is an s-cobordism between B_1 and B_2 that is a product on $\mathrm{Bd}B_1 = \mathrm{Bd}B_2$. By the s-cobordism theorem [1], [8], $W_{k-1} \cong B_1 \times [1, 2]$, the product structure extending that over $\mathrm{Bd}B_1$. Hence $\mathrm{id}\colon \mathrm{Bd}B_2 \to \mathrm{Bd}B_1$ extends to a diffeomorphism $h_{k-1}\colon M_2 \times T^{k-1} \to M_1 \times T^{k-1}$ that is the identity on $\mathrm{Cl}(N - N_{k-1}) \times T^{k-1}$. Passing to covering spaces we obtain a diffeomorphism $\tilde{h}_{k-1}\colon M_2 \times T^{k-2} \times \mathbb{R} \to M_1 \times T^{k-2} \times \mathbb{R}$ that is the identity on $\mathrm{Cl}(N - N_{k-1}) \times T^{k-2} \times \mathbb{R}$. The inductive argument is now clear.

THEOREM 4. *There exist a smooth n-manifold M and a map $g\colon (M, \mathrm{Bd}M) \to (N, \mathrm{Bd}N)$ such that*

(a) *$g|g^{-1}(N - K)$ is a diffeomorphism, and*
(b) *$g^{-1}(x)$ is cell-like for each $x \in K \cap N$.*

Proof. Applying Theorems 2 and 3, we obtain a sequence $N_0 = N \supset N_1 \supset N_2 \supset \cdots$ of neighborhoods of $K \cap N$ with smooth boundaries such that $K \cap N = \bigcap_{i=1}^{\infty} N_i$, smooth manifolds M_i $(i = 1, 2, \ldots)$ such that

(1) $M_i = (N - \operatorname{Int} N_i) \bigcup_{\operatorname{Bd} N_i} M_i$,

(2) $M_i \times T^{k-1} \times \mathbb{R}$ is diffeomorphic to $N \times T^{k-1} \times \mathbb{R}$ via a diffeomorphism whose restriction to $(N - \operatorname{Int} N_i) \times T^{k-1} \times \mathbb{R}$ is the identity, and diffeomorphisms

$$\varphi_i : M_{i+1} \to M_i$$

such that $\varphi_i | N - \operatorname{Int} N_{i-1} = \text{identity}$.

We now set $M = M_1$ and define $g^{-1}: (N, \operatorname{Bd} N) \to (M, \operatorname{Bd} M)$ as follows:

If $x \in N - \operatorname{Int} N_i$ for some i, set $g^{-1}(x) = \varphi_1 \cdots \varphi_i(x)$.

Suppose $x \in K$. Let $P_1, P_2, \ldots$ be the sequence of components of $N_1, N_2, \ldots$, respectively, that contain x. For each $i = 1, 2, \ldots$, the composition

$$M_i \to M_i \times T^{k-1} \times \mathbb{R} \to N \times T^{k-1} \times \mathbb{R} \to N,$$

when restricted to M_i', defines a homotopy equivalence $M_i' \to N_i$. Let Q_i be the component of M_i corresponding to P_i by this equivalence. Then we set $g^{-1}(x) = \bigcap_{i=1}^{\infty} \varphi_1 \cdots \varphi_i(Q_{i+1})$. Since (without loss of generality) each component of N_i is null-homotopic in N_{i-1}, $g^{-1}(x)$ is cell-like for each $x \in K$ [7].

The proof of Theorem 1 now follows immediate from Theorem 4.

Given an upper semi-continuous decomposition $\mathcal{G}$ of a space M, let $P_{\mathcal{G}}$ denote the closure of the union of the non-degenerate elements of $\mathcal{G}$ and let p: $M \to M/\mathcal{G}$ be the projection.

COROLLARY. *Suppose that G is an upper semi-continuous decomposition of an n-manifold* $M(n \geq 5)$ *such that* $p(P_{\mathcal{G}})$ *is 0-dimensional in* $M/\mathcal{G}$ *and* $M/\mathcal{G} \times \mathbb{R}^k$ *is a smooth (PL) manifold. Then there exist a smooth (PL) n-manifold* M' *and an upper semi-continuous cell-like decomposition* $\mathcal{G}'$ *of* M' *such that* $M' - P_{\mathcal{G}'}$, *is homeomorphic to* $M - P_{\mathcal{G}}$ *and* $M'/\mathcal{G}'$ *is homeomorphic to* $M/\mathcal{G}$.

Remarks. It would be interesting to know to what extent Theorem 1 is true when $n = 3$ or 4. Indeed if one assumes that the Poincaré Conjecture doesn't get in the way, then the case $n = 3$ seems manageable. When $n = 4$, however, we run into the "double suspension problem". For example, suppose that the double suspension of the dodecahedral space M^3 is the 5-sphere S^5. Then the open cone over M^3 would be a space X satisfying the hypothesis of Theorem 1 (except with $n = 4$). If Theorem 1 were true for this space X, then M^3 would bound a contractible 4-manifold. But Kervaire has shown that this is not the case [4]. In other words, Theorem 1 in dimension $n = 4$ would imply that the double suspension of the dodecahedral space is not S^5.

REFERENCES

1. D. BARDEN: The structure of manifolds. Ph.D. thesis, Cambridge University (1963).
2. H. BASS, A. HELLER and R. SWAN: *The Whitehead Group of a Polynomial Extension*. Publ. de L'Inst. des Hautes Etude Sci., No. 22 (1964).
3. L. C. GLASER: *On the Suspension of Homology Spheres* (Edited by N. KUIPER), pp. 8–16. Springer Lecture Notes in Mathematics No. 197 (1970).
4. M. A. KERVAIRE: Smooth homotopy spheres and their fundamental groups, *Trans. Am. math. Soc.* **144** (1969), 67–72.
5. R. C. KIRBY and L. C. SIEBENMANN: Deformation of smooth and piecewise linear structures (mimeograph notes).
6. K. W. KWUN and R. N. SZCZARBA: Product and sum theorems for Whitehead torsion, *Ann. Math.* **82** (1965), 183–190.
7. R. C. LACHER: Cell-like spaces, *Proc. Am. math. Soc.* **20** (1969), 598–602.
8. B. MAZUR: Relative neighborhoods and the theorems of Smale, *Ann. Math.* 77 (1963), 232–249.
9. J. W. MILNOR: Whitehead torsion, *Bull. Am. math. Soc.* 72 (1966), 358–426.
10. J. R. MUNKRES: *Elementary Differential Topology*. Princeton University Press (1966).
11. L. C. SIEBENMANN: The obstruction to finding a boundary for an open manifold in dimension greater than five. Ph.D. thesis, Princeton University (1965).
12. L. C. SIEBENMANN: On suspensions of homology spaces (preprint).
13. C. T. C. WALL: Finiteness conditions for *CW*-complexes, *Ann. Math.* **81** (1965), 56–69.

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